OPTIMAL COMBINATION OF CONTROL AND TRACKING

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G.S. SHELEMENT'EV
(Sverdlovsk)

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The present paper contains a solution of the problem of bringing a controlled system to the required state in the optimal way with a restriction imposed on the controlling force and with tracking of some of the phase coordinates. The optimal instant of switchover from tracking to control is determined. The problem considered is a certain minimax analog of the stochastic control problem considered in [1].

1. Formulation of the problem. Let there be a controlled object whose state in the time interval $t_0 \leq t \leq t_\beta$ is described by the differential Eq.

$$dx / dt = Ax + Bu \tag{1.1}$$

Here x is the n-dimensional phase coordinate vector of the controlled object, u is the r-dimensional vector of the controlling force, and A and B are constant matrices of the appropriate dimensionalities.

Let us consider the motion of this object under the following conditions.

1) The exact state $x(t_0) = x^\circ$ of the object is not known, but is restricted by some specified condition $x(t_0) \in G\{t_0\}$.

2) In order to determine more precisely the phase state of the object, the motion $x(t)$ is first tracked over some interval $t_0 \leq \tau \leq t_\alpha \leq t_\beta$, whereupon control begins at the instant $t = t_\alpha$. We assume here that measurements are taken of the coordinates $z_j(\tau)$, $t_0 \leq \tau \leq t_\alpha$ ($j = 1,\ldots, m$) of some m-dimensional vector $z(\tau)$ related to the phase vector $x(t)$ by Expression

$$z(\tau) = Hx(\tau) + \Delta(\tau) \tag{1.2}$$

where H is a constant matrix of order $m \times n$, and $\Delta(\tau)$ is the error (which may be of random character). The realization of the error $\Delta(\tau)$ is unknown, but has an intensity given by the prior estimate

$$\chi[\Delta(\tau)] \leqslant \nu, \quad t_0 \leqslant \tau \leqslant t_\alpha, \quad \nu = \text{const} > 0 \tag{1.3}$$

We shall assume from now on that the quantity $\chi[\Delta(\tau)]$ can be interpreted as some norm of the function $\Delta(\tau)$ (e.g. that $\chi[\Delta(\tau)] = \max \|\Delta(\tau)\|$, where $\|\Delta\|$ is the Euclidean norm of the vector Δ).

The phase coordinates $x_i(t_\alpha)$ are computed on the basis of the signal $z(\tau)$ by means of suitable solving operations [2 to 4].

3) The intensity $\varkappa[u]$ of the permissible controlling force in the interval $t_\alpha \leq t \leq t_\beta$ is bounded by some constant $\mu > 0$,

$$\varkappa[u(t)] \leqslant \mu \tag{1.4}$$

Once again, we assume here that the quantity $\varkappa[u]$ can be interpreted as the norm of some function, e.g. that

$$\left[\int_{t_\alpha}^{t_\beta} \|u(\tau)\|^2 d\tau\right]^{1/2} \leqslant \mu$$

Thus, we have divided the time interval $t_0 \le t \le t_\beta$ into two parts: the tracking interval $t_0 \le \tau \le t_\alpha$ and the control interval $t_\alpha \le t \le t_\beta$. We can now pose the problem of the combining of tracking and control to ensure the optimal final result of the process. In this connection it is interesting to find an instant $t = t_\alpha$ of switchover from tracking of the system motion to its control which will optimize a certain quality criterion (*). An example of such a criterion is the closeness $\varepsilon[x(t_\beta)]$ of the object to the specified state $x = x_*$ at the instant $t = t_\beta$ of termination of the process. One problem of this type is that of best mode of convergence of the phase point $x(t_\beta)$ to the origin $x = 0$. In this case

$$\varepsilon[x(t_\beta)] = (x_1^2(t_\beta) + \ldots + x_n^2(t_\beta))^{1/2}$$

However, there are situations which require ensuring of closeness to a specified state with respect to some of the coordinates only. Specifically, we can have

$$\varepsilon[x(t_\beta)] = (x_1^2(t_\beta) + \ldots + x_k^2(t_\beta))^{1/2} \qquad (k < n)$$

In the general case $\varepsilon[x(t_\beta)]$ is some given function.

Let us refine the formulation of the problem.

The problem of determining the coordinates $x_i(t_\alpha)$ of the object from the measurable quantities $z_j(\tau)$ (1.2), (1.3) with the minimal error is the problem of optimal tracking of a dynamic system. The latter can be formulated as follows [3 and 4].

P r o b l e m 1. We are required to find the optimal operation $\phi_i^\circ[z(\tau)]$ which computes the coordinate

$$x_i(t_\alpha) = \varphi_i^\circ[z(\tau)] + \omega_i = x_i^* + \omega_i \tag{1.5}$$

with the smallest guaranteed error ω_i. The required solving operation ϕ_i° must satisfy the condition of minimax $\min_\phi \sup_\Delta |\omega_i|$ of the error ω_i over all the possible errors Δ of the signal $z(\tau)$ and over all the permissible operations ϕ.

The upper bound $\delta_i(t_\alpha)$ of the modulus of the error ω_i, i.e. the quantity $\delta_i(t_\alpha) = \sup_\Delta |\omega_i|$ for $\chi[\Delta] \le \nu$ can be estimated in the known way [3 and 4] and expressed in terms of the quantity ν (1.3) and in terms of the norm of the operation ϕ_i° which solves Problem 1 (the tracking problem).

Thus, solution of the optimal tracking problem describes some domain $R\{t_\alpha, x^*\}$ in the phase space about the point $x^* = \{\phi_i^\circ[z(\tau)]\}$ by the instant $t = t_\alpha$. The points of this domain can be the true position of the object $x(t_\alpha)$ at this instant. According to the above, the domain in question is described by the inequalities

$$x_i^* - \delta_i \leqslant x_i \leqslant x_i^* + \delta_i \quad (i = 1, \ldots, n) \tag{1.6}$$

Moreover, we must take account of the result of the previous tracking fixes in the interval $t_0 \le \tau \le t_\alpha'$ ($t_\alpha' < t_\alpha$) and the initial restriction $x(t_0) \in G\{t_0\}$. Allowance for these conditions can be made recurrently. Let us assume that the last tracking fix before the instant $t = t_\alpha$ was taken at the instant $t = t_\alpha' < t_\alpha$, and that the domain $G\{t_\alpha'\}$ of possible values of $x(t_\alpha')$ has been determined. The domain $G\{t_\alpha'\}$ determines the domain $G\{t_\alpha | t_\alpha'\}$ of the states $x(t_\alpha)$ into which system of equations (1.1) with $u \equiv 0$ can pass from the states $x(t_\alpha') \in G\{t_\alpha'\}$. In other words, the points of the domain $G\{t_\alpha | t_\alpha'\}$ are given by Eqs.

$$x = X[t_\alpha, t_\alpha'] x(t_\alpha') \tag{1.7}$$

where $X[t, t_\alpha']$ is the fundamental matrix of system (1.1) and where $x(t_\alpha')$ belongs to $G\{t_\alpha'\}$.

*) In reality the tracking and control intervals are separated by a certain intermediate interval during which the decision to switch over to control is taken. We shall idealize the problem, however, by assuming that all the computations required for adopting this decision are carried out simultaneously, and even that switchover to control is possible at the instant $t = t_\alpha$ when tracking is terminated.

We conclude from this that the domain of possible states $x(t_\alpha)$ is the set $G\{t_\alpha\}$ which is the intersection of $G\{t_\alpha | t_\alpha'\}$ and $R\{t_\alpha\}$.

The problem of determining the domain $G\{t_\alpha\}$ must be solved in the course of realization of the process.

Let us suppose now that switchover to control has occurred at the instant $t = t_\alpha$. It is then expedient to consider the following problem.

Problem 2. Let the motion of the controlled object in the interval $t_\alpha \leq t \leq t_\beta$ be described by Eq. (1.1). Let the domain $G\{t_\alpha\}$ of possible states and restriction (1.4) on the controlling force u be given at the instant $t = t_\alpha$. We are to determine the optimal control $u(t)$ which ensures that

$$\varepsilon(t_\alpha) = \min_u \max_{x(t_\alpha)} \varepsilon[x(t_\beta)] \qquad (\varkappa[u] \leqslant \mu, \, x(t_\alpha) \text{from } G\{t_\alpha\}) \tag{1.8}$$

The solution of Problem 2, which follows from the known theory of linear systems control, will be described below. Let us assume for the present that the value of $\varepsilon(t_\alpha)$ at the instant $t = t_\alpha$ has been found. In order to decide whether switchover to control at the given instant $t = t_\alpha$ is advisable, we must also have a predicted value of $\varepsilon(t_\alpha')$ for $t_\alpha' > t_\alpha$. We shall denote the predicted quantity $\varepsilon(t_\alpha')$ computed on the basis of the tracking fixes obtained by the instant $t = t_\alpha$ by the symbol $\varepsilon(t_\alpha' | t_\alpha)$.

We shall compute this quantity on the basis of the most unfavorable situation which can be expected in future (i.e. when $t_\alpha' > t_\alpha$) on the basis of the data concerning the domain $G\{t_\alpha\}$ obtained at the instant $t = t_\alpha$. Let us inquire further into the meaning of the quantity $\varepsilon(t_\alpha' | t_\alpha)$. Let $t = t_\alpha'$ be some instant $(t_\alpha' > t_\alpha)$. We choose some fixed value $x(t_\alpha) = x^\alpha$ from $G\{t_\alpha\}$. On the basis of this state, by the instant $t = t_\alpha'$ the system not subject to control will arrive at the state

$$x^\alpha(t_\alpha') = X[t_\alpha', t_\alpha] x^\alpha$$

Solving in future the problem of tracking in the interval $t_0 \leq t \leq t_\alpha'$, we obtain in accordance with the foregoing the value

$$x^*(t_\alpha') = \{\varphi_i^0[z(\tau)]\} \qquad (t_0 \leqslant \tau \leqslant t_\alpha')$$

such that

$$|x_i^*(t_\alpha') - x_i^\alpha(t_\alpha')| \leqslant \delta(t_\alpha).$$

But the values $x^\alpha(t_\alpha')$ constitute the domain $G\{t_\alpha' | t_\alpha\}$. Thus, in predicting the future course of the process we must take account of all the points x_i^* lying in the $\{\delta_i(t_\alpha')\}$-neighborhood of the domain $G\{t_\alpha' | t_\alpha\}$. Let us denote this neighborhood by $Q\{t_\alpha' | t_\alpha\}$. We now infer that in future (when $t_\alpha' > t_\alpha$) we shall encounter only domains $R\{t_\alpha', x^*(t_\alpha')\}$, each of which is the intersection of the domain defined by the inequalities

$$|x_i^*(t_\alpha') - x_i| \leqslant \delta_i(t_\alpha')$$

with the domain $G\{t_\alpha' | t_\alpha\}$, where $x^*(t_\alpha')$ lies in $Q\{t_\alpha' | t_\alpha\}$. We must then solve Problem 2 on control in the segment $[t_\alpha', t_\beta]$ for each such domain $G\{t_\alpha', x^*(t_\alpha')\}$. Let this solution yield the quantity $\varepsilon(t_\alpha', x^*(t_\alpha'))$.

Next, we must consider the following problem.

Problem 3. We are to find the quantity $\varepsilon(t_\alpha' | t_\alpha)$ on the basis of the condition

$$\varepsilon(t_\alpha' | t_\alpha) = \sup_{x^*(t_\alpha')} \varepsilon(t_\alpha'; x^*(t_\alpha')) \quad \text{for} \quad x^*(t_\alpha') \in Q\{t_\alpha' | t_\alpha\} \tag{1.9}$$

The problem of choosing the instant $t = t_\alpha$ of switchover to control is now solved as follows. Let $t = t_\alpha$ be some instant $t_\alpha \geq 0$. Using the realized data $z(\tau)$ $(t_0 \leq \tau \leq t_\alpha)$, we solve Problem 1, determine the domain $G\{t_\alpha\}$ and, solving Problem 2, find the quantity $\varepsilon(t_\alpha)$. We then solve Problem 3 and construct the function $\varepsilon(\tau | t_\alpha)$ for all $\tau > t_\alpha$. If $\varepsilon(\tau | t_\alpha) > \varepsilon(t_\alpha)$ for all $\tau > t_\alpha$, then switchover to control should be effected at the instant $t = t_\alpha$; otherwise we are guaranteed in future (for $\tau = t_\alpha' > t_\alpha$), from encountering the most unfavorable situations only. On the other hand, if the function $\varepsilon(\tau | t_\alpha)$ in the time interval

$t_\alpha < \tau \le t_\beta$ is such that the inequality $\varepsilon(t_\alpha' | t_\alpha) \le \varepsilon(t_\alpha)$ is fulfilled at certain instants t_α' from this interval, then switchover to control can be effected prior to the instant $t = t_\alpha'$ at which the function $\varepsilon(\tau | t_\alpha)$ has a minimum; this is because even in the most unfavorable case switchover at the instant $t = t_\alpha'$ guarantees a better result than does switchover to control at the instant $t = t_\alpha$.

Next, solving Problems 1, 2 and 3 consecutively at the instant $t = t_\alpha'$, we use the signal $z(\tau)$ realized in the interval $t_0 \le \tau \le t_\beta'$ to find the instant $\tau = t_\alpha''$ corresponding to the next minimum of the function $\varepsilon(\tau | t_\alpha')$ until which tracking can proceed. We continue in this fashion until the instant $t = t_\alpha^0$ when $\varepsilon(\tau_\alpha' | t_\alpha^0) > \varepsilon(t_\alpha^0)$ for all $\tau > t_\alpha^0$.

We have thus developed an algorithm for determining the optimal instant $t = t_\alpha^0$ of switchover from tracking to control of system motion.

2. Solution of Problem 1. In order to determine the domain $G\{t_\alpha\}$ we must compute the quantities x_i^* and $\delta_i(t_\alpha)$ (1.6) characterizing the polyhedron $R\{t_\alpha\}$. Let us describe briefly the procedure for solving [4] the optimal tracking problem in the course of which these quantities are determined.

Let the control $u(\tau)$ (where $t_0 \le \tau \le t_\alpha$) in system (1.1) be identically equal to zero; let restriction (1.3) be imposed on the signal $z(\tau)$ (1.2). Assuming that the vector functions $z(\tau)$, $\Delta(\tau)$, and $y(\tau) = Hx(\tau)$ are elements $h(\tau)$ of some function space $B\{h(\tau), t_0 \le \tau \le t_\alpha\}$ in which the norm $\rho[h]$ is defined by Eq. $\rho[h] = \chi[h(\tau)]$ (1.3), we can determine in this space all the possible linear bounded operations $\phi_i[z(\tau)]$ among which the operation which computes the coordinates $x_i(t_\alpha)$ on the basis of the signal $z(\tau)$ $(t_0 \le \tau \le t_\alpha)$ is to be found. The functions $v_i(\tau)$ which generate the operations $\phi_i[h(\tau)]$ in the space B constitute the adjoint space B^* in which the norm of the functions v_i and the norm of the operations ϕ_i coincide, $\chi^*[v_i] = \chi^*[\phi_i]$. The form of the operation ϕ_i is determined each time by the choice of the space B.

For example, if the signals $z(\tau)$ form the space $C\{h(\tau)\}$ of functions which are continuous in $[t_0, t_\alpha]$ and have the norm

$$\chi[h] = \max_\tau \| h(\tau) \|$$

then the general form of the linear operation is given by the Stieltjes integral

$$\varphi[h(\tau)] = \int_{t_0}^{t_\alpha} h'(\tau)\, dV(\tau)$$

and the norm $\chi^*[\phi]$ of the operation ϕ is defined by Eq.

$$\chi^*[\varphi] = \operatorname{var}[V, \ t_0 \leqslant \tau \leqslant t_\alpha]$$

Here $V(\tau)$ is a bounded function, and $\operatorname{var}[V, t_0 \le \tau \le t_\alpha]$ is the total range of variation of the function $V(\tau)$ in the segment $[t_0, t_\alpha]$.

Let us make use of the minimax rule [4] to isolate from among the operations ϕ_i the optimal solving operation $\phi_i^\circ[z(\tau)]$ which yields the smallest absolute error ω_i in the most unfavorable case of the signal $z(\tau)$ (1.2), (1.3). To this end we choose from among the signals $y(\tau)$ those which carry the quantities $x_i(t_\alpha) = 1$.

Using the notation of [4], we obtain

$$\{y(\tau) \mid x_i(t_\alpha) = 1\} = [HX[\tau, t_\alpha]\, x(t_\alpha)]_{x(t_\alpha)=1}$$

Knowing the signals $\{y(\tau) | x_i(t_\alpha) = 1\}$, we can find the minimal signal $y^\circ(\tau)$ from the condition

$$\chi^\circ = \chi[y^\circ(\tau)] = \min_y \chi[\{y(\tau) \mid x_i(t_\alpha) = 1\}]$$

The optimal tracking problem has a solution if and only if $\chi^\circ = \chi[y^\circ(\tau)] > 0$.

According to the minimax rule, the optimal solving operation ϕ_i° has the norm $\chi^*[\phi_i^\circ[z(\tau)]] = 1/\chi^\circ$ and can be identified among the other linear operations ϕ_i by the maximum property, i.e. by the fact that on the minimal signal $y^\circ(\tau)$ this operation yields the maximum possible result as compared with all the other operations ϕ_i with the same norm $\chi^*[\phi_i] =$

$= 1/\chi^{\circ}$. Expressing this mathematically, we have

$$\varphi_i^{\circ}[y^{\circ}(\tau)] = \max_{\varphi}\{\varphi_i[y^{\circ}(\tau)] \quad \text{for} \quad \chi^*[\varphi_i] = 1/\chi^{\circ}\}$$

The quantities $\delta_i(t_\alpha)$ are given by Formula

$$\delta_i(t_\alpha) = \sup_{\Delta}|\varphi_i^{\circ}[\triangle(\tau)]| = \nu\chi^*[\varphi_i^{\circ}[z(\tau)]] = \nu/\chi^{\circ}$$

The intersection of the domains $R\{t_\alpha\}$ (1.6) and $G\{t_\alpha | t_\alpha^{\circ}\}$ (1.7) defines the required domain $G\{t_\alpha\}$ of possible states at the instant $t = t_\alpha$.

3. Solution of Problem 2. We now turn to the determination of the quantity $\varepsilon(t_\alpha)$ (1.8) characterizing the closeness of the phase point $x(t_\beta)$ to the specified state $x = x_*$ at the instant of termination of the process $t = t_\beta$.

Thus, let us assume we are given the instant $t = t_\alpha$, the domain $G\{t_\alpha\}$ of possible states of the system at this instant, and the set $P\{u : \varkappa[u] \leq \mu\}$ of permissible controls u (1.4). For each fixed control u from $P\{u\}$ the quantity $\varepsilon[x(t_\beta)]$ depends on the choice of the initial value of $x(t_\alpha)$, and the most unfavorable case, i.e. that where the phase point $x(t_\beta)$ is most distant from the specified value $x = x_*$, is given by Expression

$$\varepsilon_u(t_\alpha) = \max_{x(t_\alpha)} \varepsilon[x(t_\beta)] \quad \text{for} \quad x(t_\alpha) \text{from} G\{t_\alpha\}$$

If we are required to ensure minimal deviation of the phase point from the position $x = x_*$ at the instant $t = t_\beta$ for any initial state $x(t_\alpha)$ from $G\{t_\alpha\}$, then we must choose a control u from $P\{u\}$ which minimizes the quantity $\varepsilon_u(t_\alpha)$. The maximal guaranteed closeness $\varepsilon(t_\alpha)$ can then be determined from the condition

$$\varepsilon(t_\alpha) = \min_u \varepsilon_u(t_\alpha) = \min_u \max_{x(t_\alpha)} \varepsilon[x(t_\beta)] \tag{3.1}$$

$$\text{for} \quad u \text{from} P\{u\}, \; x(t_\alpha) \text{ из } G\{t_\alpha\}$$

In order to solve Problem (3.1) by the substitution of variables $x = y + w$ we break down system (1.1) into two subsystems:

$$dy/dt = Ay + Bu, \qquad y(t_\alpha) = y^\alpha \tag{3.2}$$

$$dw/dt = Aw, \quad w(t_\alpha) = x(t_\alpha) - y^\alpha, \quad x(t_\alpha) \in G\{t_\alpha\} \tag{3.3}$$

The point y^α is chosen to facilitate computation. For example, we can set $y^\alpha = 0$. The linear transformation

$$w = X[t_\beta, t_\alpha]\, w(t_\alpha) \tag{3.4}$$

transforms the domain $G\{t_\alpha\}$ into some domain $W\{t_\alpha\}$. The solutions $y(t_\beta)$ of system (3.2) for various u (1.4) form the attainability domain $\Gamma[y^\alpha, t_\alpha, t_\beta, \mu]$ of the process $y(t)$ by the instant $t = t_\beta$ for $y(t_\alpha) = y^\alpha$ and for $u = u(t)$ (1.4). By the Cauchy formula we have

$$y(t_\beta) = X[t_\beta, t_\alpha]\, y^\alpha + \int_{t_\alpha}^{t_\beta} X[t_\beta, \tau]\, Bu(\tau)\, d\tau \tag{3.5}$$

From (3.1), (3.4), and (3.5) we infer that

$$\varepsilon(t_\alpha) = \min_u \gamma[y(t_\beta)] \quad \text{for} \quad \varkappa[u] \leqslant \mu \tag{3.6}$$

Here

$$\gamma[y(t_\beta)] = \max_w \varepsilon[y(t_\beta) + w] \quad \text{for} \quad w \text{from} W\{t_\alpha\} \tag{3.7}$$

Problem (3.6), (3.7) consists in determining the point y_0^β from the domain $\Gamma[y^\alpha, t_\alpha, t_\beta, \mu]$ and the control $u = u^{\circ}(t)$ which minimize the function $\gamma[y(t_\beta)]$ under condition (1.4). In other words, we must determine the points $y(t_\beta)$ which form the attainability domain of the process $y(t)$. To find the attainability domain $\Gamma[y^\alpha, t_\alpha, t_\beta, \mu]$ of the process let us consider the problem of optimal transfer [3] of system (3.2) from the initial point y^α to some temporarily fixed point y^β in a time $t_\alpha \leq t \leq t_\beta$ under the condition of minimal intensity $\varkappa[u]$. As we know [4], the solution of such a problem can be reduced to finding the vector k which solves the problem

$$\max_k c'[y^\beta]k = \zeta^\circ[y^\beta] \tag{3.8}$$

under the condition

$$\rho[B'S[\tau, t_\beta]k] \leqslant 1 \tag{3.9}$$

where $S[t, t_\beta]$ is the fundamental matrix of the system $ds/dt = -A's$ adjoint to system (3.3), and where $c[y^\beta] = y^\beta - X[t_\beta, t_\alpha]y^\alpha$. The control $u^\circ(t)$ which solves the problem of optimal transfer of system (3.2) from the position y^α to the position y^β has the norm $\rho^*[u] = \zeta^\circ[y^\beta]$ and can be determined from the maximum rule [4],

$$\int_{t_\alpha}^{t_\beta} k^{\circ\prime}S[\tau, t_\beta]Bu^\circ(\tau)\,d\tau = \max_u \int_{t_\alpha}^{t_\beta} k^{\circ\prime}S[\tau, t_\beta]Bu(\tau)\,d\tau \tag{3.10}$$

$$\text{for}\quad \rho^*[u] \leqslant \zeta^\circ[y^\beta]$$

where k° and $\zeta^\circ[y^\beta]$ are the solution of problem (3.8), (3.9).

Thus, by solving problem (3.8), (3.9) we obtain an expression for the control intensity in the form of the function $\zeta^\circ[y^\beta]$ of the final state $y(t_\beta)$ of system (3.2) at the instant $t = t_\beta$. In view of the fact that the intensity is bounded by the constant μ (1.4), we infer from (3.6) and (3.7) that the problem of determining $\varepsilon(t_\alpha)$ reduces to the problem of finding the arbitrary extremum

$$\min_y \gamma[y(t_\beta)] = \varepsilon(t_\alpha) \tag{3.11}$$

under the condition

$$\zeta^\circ[y(t_\beta)] \leqslant \mu \tag{3.12}$$

where $\zeta^\circ[y(t_\beta)]$ must be determined from conditions (3.8), (3.9).

Having determined the point y_0^β corresponding to the minimum from (3.7), (3.11), and (3.12), we can find from (3.8) and (3.9) the optimal control $u^\circ(t)$ which ensures maximal closeness $\varepsilon(t_\alpha)$ of the phase point to the specified state $x = x_*$.

N o t e s 3.1. We note that in those cases where problem (3.6), (3.7) involves minimization of the function $\gamma[y(t_\beta)]$ whose datum levels $\gamma[y(t_\beta)] = \text{const}$ are convex, the problem of determining the value of $\varepsilon(t_\alpha)$ becomes simpler, since some of the minimization and maximization operations in problem (3.8) to (3.10) can then be interchanged [3, 5 and 6].

3.2. We have described a procedure for determining the optimal instant t_α° of switchover from tracking to control of object motion under the assumption that the tracking problem is solved each time at the instant $\tau = t_\alpha'$ when the function $\varepsilon(\tau \mid t_\alpha)$ assumes its minimal value in the segment $[t_\alpha', t_\beta]$. It is sometimes convenient to follow a similar procedure in which the sequence of instants $t_k = t_{k-1} + \Delta t$ of the tracking fixes is preselected rather than chosen on the basis of the minimum condition for the function $\varepsilon(\tau \mid t_\alpha)$.

4. Example. Let us consider a material point whose motion along the straight line ξ is described by Eqs.

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = 0, \quad 0 \leqslant t \leqslant 1 \qquad \left(x_1 = \xi,\ x_2 = \frac{d\xi}{dt}\right) \tag{4.1}$$

We assume that the exact velocity of the point at $t = 0$ is not known, but that the velocity at this instant satisfies the condition $m_0 \leq x_2(0) \leq n_0$. The velocity $x_2(t)$ at the instant $\tau = t$ can be determined by measuring the coordinate x_1. This measurement involves some error $\omega_1(t)$ of bounded magnitude

$$|\omega_1(t)| \leqslant \delta, \qquad \delta > 0 - \text{const} \tag{4.2}$$

We also assume that the motion of the point can be corrected by varying the velocity of the point, but that the supply of energy $\varkappa[u]$ available for this correction is limited,

$$\varkappa[u] = \left[\int_{t_\alpha}^{t_\beta} u^2(\tau)\,d\tau\right]^{1/2} \leqslant \mu, \quad \mu > 0 - \text{const} \tag{4.3}$$

We are required to choose the instant $t = t_\alpha^\circ$ of switchover from tracking to control in such a way that in the time $(1 - t_\alpha^\circ)$ remaining for control, the control u (4.3) can be used to minimize the velocity of the point, i.e. such that

$$\varepsilon [x(t_\beta)] = |x_2(t_\beta)| = \min_{t_\alpha} \qquad (4.4)$$

at the instant $t = t_\beta$ of termination of motion.

We know [4] that the optimal solving operation $\phi^\circ[x_1]$ which determines the velocity $x_2(t_\alpha)$ of a point moving by inertia ($u(\tau) \equiv 0$, $0 \le \tau \le t_\alpha$) at the instant $t = t_\alpha$ under condition (4.2) is given by

$$\varphi^\circ [x_1] = [x_1(t_\alpha) - x_1(0)] / t_\alpha = x_2(t_\alpha) \qquad (4.5)$$

and therefore coincides with the standard formula for computing the velocity of a uniformly moving point. We note that the optimal solving operation $\phi^\circ[x_1]$ has a form different from (4.5) for a different specified intensity of the error $\Delta(\tau)$. The velocity $x_2(t_\alpha)$ is computed with the error $\omega_2(t_\alpha)$, and $|\omega_2(t_\alpha)| \le 2\delta/t_\alpha$.

The domain of possible states $G\{t_{\alpha i}\}$ is a segment $[n_i, m_i]$, where

$$n_i = \begin{cases} n_{i-1} & \text{for } y_{\alpha i} + \Lambda_i \geqslant n_{i-1} \\ y_{\alpha i} + \Lambda_i & \text{for } y_{\alpha i} + \Lambda_i < n_{i-1} \end{cases}, \qquad m_i = \begin{cases} m_{i-1} & \text{for } y_{\alpha i} - \Lambda_i \leqslant m_{i-1} \\ y_{\alpha i} - \Lambda_i & \text{for } y_{\alpha i} - \Lambda_i > m_{i-1} \end{cases}$$

Here $y_{\alpha i} = x_2(t_{\alpha i})$ (1.5), (4.5) and $\Lambda_i = 2\delta/t_{\alpha i}$, and the instants $t_{\alpha i}$ are found at the $(i-1)$-th step of solving Problem 3. Recalling that the maximum of the function $\varepsilon[x(t_\beta)]$ with respect to w is attained on the boundary of the domain $W\{t_{\alpha i}\}$ (3.4), i.e. at the point $w_{\alpha i} = (n_i - m_i)/2$, we find that at each instant $t = t_{\alpha i}$ the function $\gamma[y]$ (3.7) is given by

$$\gamma[y] = |y| + w_{\alpha i} \qquad (4.6)$$

and that the function $\zeta^\circ[y]$ after we have solved problem (3.8), (3.9), (4.3) is given by Expression

$$\zeta^\circ[y] = \frac{\mu\,|y - y_i^\circ|}{M(t_{\alpha i})}, \quad y_i^\circ = \frac{n_i + m_i}{2}, \quad M(t_{\alpha i}) = \mu(1 - t_{\alpha i})^{1/2} \qquad (4.7)$$

We have thus reduced the problem to finding the minimum of the function (4.6) under Condition (4.7). At each instant $t = t_{\alpha i}$ the minimum $\varepsilon(t_\alpha)$ (3.11) of the function $\gamma[y]$ (4.6) either: a) equals $w_{\alpha i}$ and is attained at the point $y = 0$ (if $|y_{\alpha i}| \le M(t_{\alpha i})$), or b) equals $w_{\alpha i} + |y_{\alpha i}| - M(t_{\alpha i})$ and is attained at one of the ends of the segment $[y_{\alpha i} - M(t_{\alpha i})\, y_{\alpha i} + M(t_{\alpha i})]$ (if $|y_{\alpha i}| > M(t_{\alpha i})$). In the latter case switchover to control should be effected immediately.

The predicting function $\varepsilon(\tau\,|\,t_{\alpha i})$ for all $\tau > t_{\alpha i}$ is given by

$$\varepsilon(\tau\,|\,t_{\alpha i}) = \begin{cases} w_{\tau i} & \text{for } |y_{\tau i}| \leqslant M(\tau) \\ w_{\tau i} + |y_{\tau i}| - M(\tau) & \text{for } |y_{\tau i}| > M(\tau) \end{cases}$$

where $w_{\tau i}$, $t_{\tau i}$, and $M(\tau)$ denote the quantities

$$w_{\tau i} = \begin{cases} \Lambda & \text{for } \Lambda \leqslant w_{\alpha i}, \\ w_{\alpha i} & \text{for } \Lambda > w_{\alpha i}, \end{cases} \qquad y_{\tau i} = \begin{cases} (|N_i| - \Lambda)\,\mathrm{sign}\,N_i & \text{for } \Lambda \leqslant w_{\alpha i} \\ y_i^\circ & \text{for } \Lambda > w_{\alpha i} \end{cases}$$

$$N_i = \begin{cases} n_i & \text{for } |n_i| \geqslant |m_i| \\ m_i & \text{for } |n_i| < |m_i| \end{cases} \qquad \Lambda = \frac{2\delta}{\tau}, \qquad M(\tau) = \mu(1 - \tau)^{1/2}$$

The above procedure was realized on a computer for various values of the constants δ, μ, m_0, and n_0, and specifically (see Fig. 1) for $\delta = 0.1$, $\mu = 3.5$, $n_0 = -m_0 = 3$, $x_2^* = 1$. The instants of prediction turned out to be $t_{\alpha 1} = 0.452$, $t_{\alpha 2} = 0.867$, $t_{\alpha 3} = 0.902$. For $t = t_{\alpha 3}$ we obtained $\varepsilon(\tau\,|\,t_{\alpha 3}) > \varepsilon(t_{\alpha 3}) = 0.137$ for all $\tau > t_{\alpha 3}$. Hence, the instant $t = t_{\alpha 3}$ was the optimal instant of switchover from tracking to control of motion of the point.

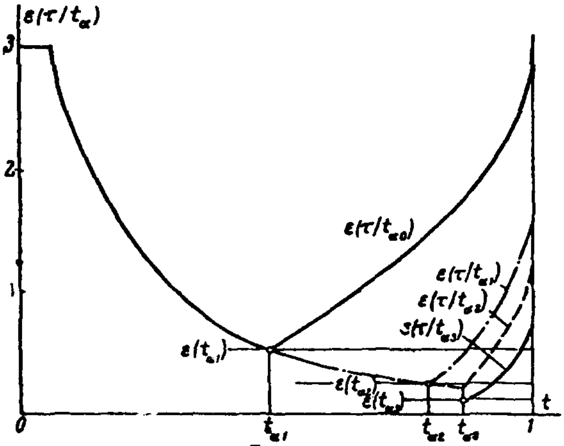

Fig. 1

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